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Geometry on Ruled Surfaces.

By S. Lefschetz.

1. The object of this paper is to prove a formula given by W. E. Story for the number of common points of two curves C_a and C_b of order a and b lying on a scroll S_μ of order μ. He stated that

$$[C_a C_b] = b\alpha + a\beta - \mu\alpha\beta, \tag{1}$$

where $[C_a C_b]$ designates the number of intersections of the two curves C_a and C_b, α and β the number of intersections of C_a and C_b with an arbitrary generator S_μ. He gave the proof himself for $\mu=2$, 3, and F. B. Williams gave it for $\mu=4$, 5.* Of the two proofs here given, one is based upon a classical work of Severi,† the other is of a very elementary character.

2. Before proceeding with our first proof, let us recall a few definitions introduced by Severi. Two curves C and D traced on a surface S will be called *equivalent*, and we shall write $C \equiv D$, if there exists on S an algebraic system of curves containing them both totally. From this definition it is easy to pass to the meaning of $\lambda C \equiv \lambda' D$, λ, λ' being both positive integers, and finally to the meaning of $\Sigma \lambda_i C_i \equiv 0$, the λ's being integers. Severi proves that we can find on S, ρ curves such that between them and any other curve C on S there is always a relation of the type

$$\lambda C + \sum_1^\rho {}_i \lambda_i C_i \equiv 0,$$

The numbers $(\lambda, \lambda_1, \ldots, \lambda_\rho)$ are the *characters of C with respect to the base* $(C_1, C_2, \ldots, C_\rho)$. He also gives the so-called Bezout Theorem for a surface by showing that if for D we have

$$\mu D + \sum_1^\rho \mu_i C_i \equiv 0,$$

then

$$[CD] = \frac{1}{\lambda\mu} \sum_1^\rho {}_{i,k} \lambda_i \mu_k [C_i C_k].$$

In particular he states this for a scroll $\rho = 2$. We shall prove this here, by

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[*] "Curves on Quintic Scrolls," American Journal of Mathematics, Vol. XXXIV (1912), p. 421. A complete bibliography is given there.

[†] "Sulla totalita delle curve algebriche, *Math. Ann.*, Vol. LXII, p. 124.

showing that, for S_μ, a base is found taking for C_1 a generator and for C_2 a plane section.*

3. Let $a=\mu\nu-h$, $h<\mu$, and let k be a positive integer. A surface of order $k\nu$ going through k curves infinitely near C_a, that is to say, through $k\,C_a$, has to satisfy at most $k(ak\nu+1)$ conditions; and if it has to go through kh arbitrary generators of S_μ, it must satisfy at most $kh(k\nu-k\alpha+1)$ conditions, so that the number of linearly independent surfaces of order $k\nu$ going through kC_a, kh generators, and not containing S_μ as a part, is not inferior to

$$N_{k\nu}=\binom{k\nu+3}{3}-\binom{k\nu-\mu+3}{3}-k(\alpha k\nu+1)-kh(k\nu-k\alpha+1)$$
$$=[\mu\nu^2\frac{(\mu-2)}{2}+\alpha h]k^2+\ldots.,$$

the unwritten terms containing k to a power lower than 2. The coefficient of k^2 is positive if $\mu>2$. Hence, *when k is above a certain limit, kC_a has kh generators for residual intersection.* It is well known that for $\mu=2$, any curve is either a complete intersection or has a generator for residual intersection, so that the above is true without any restrictions. In the notation of paragraph 2, we have, therefore,

$$kC_a+khC_1-k\nu C_2\equiv 0.$$

As kC_a and khC_1 are cut out by a surface of order $k\nu$ and two arbitrary generators have no common points, it follows that the $k\nu$ intersections of a generator with the $S_{k\nu}$ will be on kC_a, so that

$$k\alpha=k\nu;\quad \therefore\ \alpha=\nu,$$

showing that the characters of C_a with respect to the base (C_1, C_2) are $(k, kh, -k\alpha)$. Similarly, C_b will have characters $(k', k'h', -k'\beta)$, with $a=\alpha\mu-h$, $b=\beta\mu-h'$. We also have

$$[C_1C_1]=0,\quad [C_1C_2]=1,\quad [C_2C_2]=\mu;$$

hence, by Bezout's theorem for S_μ,

$$[C_aC_b]=-h\beta-h'\alpha+\mu\alpha\beta=(a-\alpha\mu)\beta+(b-\beta\mu)\alpha+\mu\alpha\beta=a\beta+b\alpha-\mu\alpha\beta,$$

which was to be proved.

4. The following proof is elementary. The coordinates of any point on a generator can be expressed thus:

$$x_i=m_i-\lambda n_i,\qquad (i=1, 2, 3, 4),$$

where $m_i=m_i(\xi_1, \xi_2, \xi_3)$ and $n_i=n_i(\xi_1, \xi_2, \xi_3)$ are homogeneous functions of the ξ's of the same order ρ, λ is an arbitrary parameter, while the ξ's satisfy a relation $f(\xi_1, \xi_2, \xi_3)=0$, representing in their plane a curve of order m and same genus p as S_μ. If a curve C_a meets an arbitrary generator in α points, it will be represented by an equation of the type

$$\phi_0\lambda^\alpha+\phi_1\lambda^{\alpha-1}+\phi_2\lambda^{\alpha-2}+\ldots.+\phi_\alpha=0,$$

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*In the paper previously cited, Severi refers the reader, for a proof, to his paper, "Sulle corrispondenze tra i punti," *Torino Memorie,* Vol. LXIV. This being of difficult access, we give below our very simple proof for the sake of completeness.

the ϕ's being all homogeneous polynomials of same order a' in the ξ's. The curves $\phi_i=0$ will have in the ξ-plane a certain number α' of points in common with $f=0$. To find the order a of C_a, we have to find the number of its intersections with a variable plane section:

$$\sum_1^4 A_j m_j-\lambda\sum_1^4 A_j n_j=0.$$

The result of the elimination of λ between the equations of the two curves is:

$$\sum_0^a{}_i\,\phi_i\Big(\sum A_j m_j\Big)^{a-i}\Big(\sum A_j n_j\Big)^i=0,$$

of which we must find the number of variable intersections with $f=0$. If the curves $f=0$, $m_i=0$, $n_i=0$, $(i=1, 2, 3, 4)$, have γ common points, we have

$$a=(a'+\rho\alpha)m-\alpha\gamma.$$

If the equation

$$\psi_0\lambda^\beta+\psi_1\lambda^{\beta-1}+\ldots\ldots+\psi_\beta=0,$$

where the ψ's are of order b' in the ξ's, represents a second curve C_b of order b, we have

$$b=(b'+\rho\beta)m-\beta\gamma.$$

We obtain $[C_aC_b]$ by eliminating λ between the equations of the two curves. This gives:

$$R(\xi_1,\xi_2,\xi_3)=\left|\begin{array}{cccc}\phi_0&\phi_1&\phi_2&\ldots\ldots\\ 0&\phi_0&\phi_1&\ldots\ldots\\ 0&0&\phi_0&\ldots\ldots\\ 0&0&0&\ldots\ldots\\ \psi_0&\psi_1&\psi_2&\ldots\ldots\\ 0&\psi_0&\psi_1&\ldots\ldots\\ 0&0&\psi_0&\ldots\ldots\\ 0&0&0&\ldots\ldots\end{array}\right|\begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\beta\\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\}\alpha\end{array}=0,$$

which represents in the ξ-plane a curve of order $(\alpha b'+\beta a')$ intersecting $f=0$ in $m(\alpha b'+\beta a')$ points. From these we must deduce the points common to all the ϕ's or the ψ's and f. A point of multiplicity 9 for all the ϕ's and multiplicity r for f will count for $9r$ points in the intersection of f and ϕ_i, and for $\beta 9r$ points in the intersection of f with R. A similar remark applies to the ψ's, so that finally:

$$\begin{aligned}[C_aC_b]&=m(\beta a'+\alpha b')-\alpha'\beta-\beta'\alpha\\ &=\beta(ma'-\alpha')+\alpha(mb'-\beta')\\ &=a\beta+b\alpha-2(m\rho-\gamma)\alpha\beta.\end{aligned}$$

In particular, if C_a and C_b are two plane sections of S_μ, we have $a=b=\mu$, $\alpha=\beta=1$.

$\therefore$ $[C_aC_b]=\mu=2\mu-2(m\rho-\gamma)$; $\therefore$ $2(m\rho-\gamma)=\mu$; $\therefore$ $[C_aC_b]=b\alpha+a\beta-\mu\alpha\beta$, and the proof is complete.

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